

Instability of a shallow-water potential-vorticity front

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(Received 1 August 2005 and in revised form 12 January 2006)

A straight front separating two semi-infinite regions of uniform potential vorticity (PV) in a rotating shallow-water fluid gives rise to a localized fluid jet and a geostrophically balanced shelf in the free surface. The linear stability of this configuration, consisting of the simplest non-trivial PV distribution, has been studied previously, with ambiguous results. We revisit the problem and show that the flow is weakly unstable when the maximum Rossby number $R > 1$. The instability is surprisingly weak, indeed exponentially so, scaling like $\exp[-4.3/(R-1)]$ as $R \rightarrow 1$. Even when $R = \sqrt{2}$ (when the maximum Froude number $F = 1$), the maximum growth rate is only 7.76×10^{-6} times the Coriolis frequency. Its existence nonetheless sheds light on the concept of ‘balance’ in geophysical flows, i.e. the degree to which the PV controls the dynamical evolution of these flows.

1. Introduction

Two nearly distinct types of motion are found in the Earth's atmosphere and oceans, namely ‘balanced’ vortical motions and ‘unbalanced’ gravity-wave motions. The balanced motions are controlled entirely by a materially advected scalar, the potential vorticity (PV), from which all other dynamical fields (velocity, pressure, etc.) can be derived via prescribed ‘inversion relations’. The residual motions are classified as unbalanced motions, and are presumed to be gravity waves.

This decomposition is only strictly defined, however, for the linearized equations about a state of rest. Otherwise, a degree of ambiguity arises (surrounding the choice of inversion relations, for instance), making it impossible to define uniquely the balanced part of the flow (cf. Mohebalhojeh & Dritschel 2001; Viúdez & Dritschel 2004 and references therein). Nevertheless, such a decomposition, even if inexact, is often of great practical use, particularly in weather forecasting. The ambiguity in the definition of balance can be exceedingly small in many circumstances.

A long-standing problem in geophysical fluid dynamics concerns the quantification of the coupling between the two types of motion and, in particular, the mechanisms for the generation of gravity waves by balanced motion (e.g. Lorenz & Krishnamurthy 1987; Warn 1997; Ford, McIntyre & Norton 2000; Vanneste 2004 and references therein). Among these mechanisms, unbalanced instabilities of steady (balanced) flows have recently received a great deal of attention (e.g. Ford 1994; Yavneh, McWilliams & Molemaker 2001; McWilliams, Molemaker & Yavneh 2004; Plougonven, Muraki & Snyder 2005). These instabilities have several significant features: with their growing

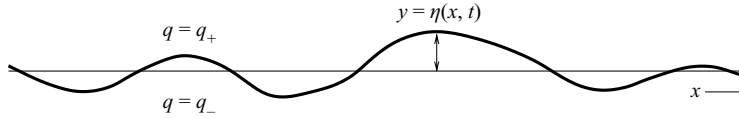


FIGURE 1. Schematic diagram of a disturbed PV front. In the linear stability analysis, the displacement η is taken to be purely sinusoidal.

modes of mixed nature, they reveal, in a simple context, the coupling that exists between balanced and unbalanced motion in a basic state flow not at rest, and it has been suggested that they may contribute to energy dissipation of balanced flows (McWilliams, Molemaker & Yavneh 2001).

Here, we focus, or rather refocus, on the simplest instability mechanism of this type. We consider a shallow-water flow, rotating uniformly at rate $f/2$, on an infinite plane, and examine the linear stability of the simplest non-trivial PV distribution, namely that of a single-step discontinuity along a front $y = \eta(x, t)$, whose undisturbed position is $y = 0$ (see figure 1). This configuration is characterized by a single non-dimensional number, which we take to be the maximum Rossby number R or, equivalently, the maximum Froude number F . It admits a single balanced or vortical wave, known as a Rossby wave, which predominantly displaces the interface, and an infinite set of gravity waves, just as in the case of a basic state at rest.

The stability of this PV-front was first examined, for a special case, by Paldor (1983), then in the general case studied here by Ford (1993). Ford applied Ripa's (1983) theorem to prove that the front is formally (and hence linearly) stable for $R < 1$, i.e. $F < 1/\sqrt{2}$. He also established the existence of a linear instability mechanism for $R > 1$ ($F > 1/\sqrt{2}$) whose growth rate is exponentially small in the limit of large x -wavenumber $k \gg 1$. Boss, Paldor & Thompson (1996) dismissed this instability on the grounds that the disturbance mode had unbounded energy. This is not the case. The unstable modes decay exponentially far from the front, and hence have finite energy. We clarify this point in this paper.

We extend Ford's (1993) numerical and analytical results by examining the instability in detail in the regime $F < 1$, especially near the threshold of instability. In this regime, the observed growth rates, some 5–6 orders of magnitude smaller than f , require highly accurate numerics to capture. Ford's WKB analysis is also extended in order to obtain asymptotic estimates for the maximum growth rates and threshold wavenumber as $R \rightarrow 1$ ($F \rightarrow 1/\sqrt{2}$). The maximum growth rate is found to be achieved for a wavenumber that is asymptotically close to the threshold wavenumber; as a result, it is several orders of magnitude smaller than might have been anticipated from more straightforward asymptotics. We also demonstrate that some properties of the instability can be predicted with remarkable accuracy using a quasi-geostrophic description of the Rossby wave, in spite of the fact that $R \geq 1$.

The observed instability points to a breakdown of the concept of balance in a linear system when $R > 1$. This is probably the simplest fluid-dynamical context in which this occurs. The extremely small growth rates indicate that balance may dominate even in parameter regimes where there is no frequency separation between balanced and unbalanced motions. This finding provides theoretical support to the observation that rotating flows are often close to some form of balance even in the absence of a frequency separation.

The structure of the paper is as follows. The next section presents the governing equations, the basic state, the linear system to be analysed, and the solution procedure.

This is followed in §§3 and 4 by numerical results and the asymptotic analysis, including comparisons and careful checks on accuracy. The paper concludes in §5 with some implications and a discussion of the nonlinear problem.

2. Formulation

2.1. Mathematical model

We consider an unbounded inviscid rotating shallow fluid layer held down by gravity. Its evolution is modelled by the shallow-water equations

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - fv = -\frac{\partial \phi}{\partial x}, \tag{2.1}$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + fu = -\frac{\partial \phi}{\partial y}, \tag{2.2}$$

$$\frac{\partial \phi}{\partial t} + \frac{\partial(u\phi)}{\partial x} + \frac{\partial(v\phi)}{\partial y} = 0, \tag{2.3}$$

where f is the Coriolis parameter (twice the background rotation rate), u and v are the x and y velocity components, and $\phi = gh$ is the geopotential (cf. Gill 1982).

The above equations can be combined to prove that the potential vorticity

$$q = \frac{\partial v/\partial x - \partial u/\partial y + f}{\phi}, \tag{2.4}$$

is materially conserved following fluid particles. In other words, contours of constant q move with the fluid. The simplest non-trivial PV distribution imaginable consists of a single PV jump or front separating two regions of uniform q in the plane, with $q = q_{\pm}$ above and below the front $y = \eta(x, t)$ (see figure 1). The conservation of q then reduces to the fact that the front is a material line.

The basic flow that we consider corresponds to a straight front at $y=0$; it is characterized by a shelf $\phi = \bar{\phi}(y)$ in geostrophic balance with a zonal jet $(u, v) = (\bar{u}(y), 0)$. Without loss of generality, we can take $f = 1$ and $c_+c_- = 1$, where $c_{\pm} = [\bar{\phi}(\pm\infty)]^{1/2}$ are the short-scale gravity-wave speeds far from the front. We then obtain

$$\bar{\phi}(y) = c_{\pm}^2 \pm Fc_{\pm}e^{-|y|/c_{\pm}}, \quad \bar{u}(y) = -\frac{d\bar{\phi}(y)}{dy} = Fe^{-|y|/c_{\pm}}, \tag{2.5}$$

where we have introduced the maximum Froude number $F = c_- - c_+$, which we use as control parameter for the basic state flow. In terms of F , $c_{\pm} = (1 + F^2/4)^{1/2} \mp F/2$. As an alternative to F , we also use the maximum Rossby number R which satisfies $F = R/\sqrt{1 + R}$. (Ford (1993) used the ratio $q_+/q_- = (1 + R)^2$.)

Ford (1993) applied Ripa's (1983) theorem to establish that the basic flow (2.5) is stable for $F < 1/\sqrt{2}$, i.e. $R < 1$. On the other hand, one expects that for $F > 1$, the flow is susceptible to the formation of shocks which violate the hydrostatic approximation used to derive the shallow-water equations. We therefore restrict our attention to the regime $1/\sqrt{2} \leq F \leq 1$.

The linear stability of (2.5) is addressed by adding infinitesimal disturbances of the form $\{\hat{u}(y), \hat{v}(y), \hat{\phi}(y)\}e^{i(kx - \sigma t)}$ and linearizing. Here k is the disturbance wavenumber and σ is the disturbance frequency (a positive imaginary part implying instability). We seek disturbances which do not change the basic-state PV except through displacements of the front. This amounts to replacing any one of the three equations

(2.1)–(2.3) with

$$q\hat{\phi} = ik\hat{v} - \hat{u}', \quad (2.6)$$

where $q = q_{\pm} = 1/c_{\pm}^2$ for $y \gtrless 0$ and the prime denotes d/dy . We obtain two other independent equations using (2.1) and (2.3),

$$i(k\bar{u} - \sigma)\hat{u} + (\bar{u}' - 1)\hat{v} + ik\hat{\phi} = 0, \quad (2.7)$$

$$i(k\bar{u} - \sigma)\hat{\phi} + ik\bar{\phi}\hat{u} + (\bar{\phi}\hat{v})' = 0. \quad (2.8)$$

Subtracting the limits of (2.7) as $y \rightarrow 0^{\pm}$ and imposing continuity of \hat{v} and $\hat{\phi}$ gives the jump condition

$$i(kF - \sigma)[\hat{u}(0^+) - \hat{u}(0^-)] = (c_+ + c_-)F\hat{v}(0). \quad (2.9)$$

Together with the requirement that all fields vanish as $y \rightarrow \pm\infty$, (2.6)–(2.9) form an eigenvalue problem for σ with k and F as sole parameters.

2.2. Numerical considerations

The solution procedure follows in part Paldor (1983) and Boss *et al.* (1996), with some important differences. First, we substitute $\hat{w} = i\hat{v}$ to factor out the dependence on i . As the non-constant coefficients in (2.6)–(2.8) involve only exponential functions, we seek solutions of the form

$$(\hat{u}/c_{\pm}, \hat{w}/c_{\pm}, \hat{\phi}/c_{\pm}^2) = e^{-K^{\pm}|y|} \sum_{n=0}^{\infty} (\hat{u}_n^{\pm}, \hat{w}_n^{\pm}, \hat{\phi}_n^{\pm}) e^{-n|y|/c_{\pm}}. \quad (2.10)$$

Then, equating coefficients of $e^{-\Omega_n^{\pm}|y|/c_{\pm}}$ for each n , where $\Omega_n^{\pm} \equiv c_{\pm}K^{\pm} + n$, we obtain the recurrence relations

$$k_{\pm}\hat{w}_n^{\pm} = \hat{\phi}_n^{\pm} \mp \Omega_n^{\pm}\hat{u}_n^{\pm}, \quad (2.11)$$

$$k_{\pm}\hat{\phi}_n^{\pm} - \sigma\hat{u}_n^{\pm} + \hat{w}_n^{\pm} = -\varepsilon_{\pm}(k_{\pm}\hat{u}_{n-1}^{\pm} \pm \hat{w}_{n-1}^{\pm}), \quad (2.12)$$

$$k_{\pm}\hat{u}_n^{\pm} - \sigma\hat{\phi}_n^{\pm} \pm \Omega_n^{\pm}\hat{w}_n^{\pm} = -\varepsilon_{\pm}(k_{\pm}\hat{\phi}_{n-1}^{\pm} \pm k_{\pm}\hat{u}_{n-1}^{\pm} + \Omega_n^{\pm}\hat{w}_{n-1}^{\pm}), \quad (2.13)$$

where $k_{\pm} \equiv kc_{\pm}$ are scaled wavenumbers, and $\varepsilon_{\pm} \equiv F/c_{\pm}$ are the local Rossby numbers at $y = 0^{\pm}$ (the larger being $\varepsilon_+ = R$). When $n = 0$, the $n - 1$ terms are absent, and solvability (for $k \neq 0$) requires that

$$c_{\pm}K^{\pm} = \sqrt{k_{\pm}^2 + 1 - \sigma^2} \quad (2.14)$$

and $\hat{\phi}_0^{\pm} = \hat{u}_0^{\pm}(\sigma k_{\pm} \pm c_{\pm}K^{\pm})/(k_{\pm}^2 + 1)$. Note that $K^{\pm} = 0$ gives the dispersion relation for inertia-gravity waves far from the front.

We start with a guess for σ , using the small F ‘quasi-geostrophic’ approximation

$$\sigma = kF[1 - (1 + k^2)^{-1/2}] + O(F^2) \quad (2.15)$$

(cf. Nycander, Dritschel & Sutyrin 1991). This turns out to be an excellent approximation up to $F = 2$. Taking $\hat{u}_0^{\pm} = 1$ arbitrarily, (2.11)–(2.13) are solved recursively for the higher-order coefficients. Here, we use 1000 coefficients, sufficient to ensure machine precision (at quadruple precision) except when $R \gtrsim 1$, see below. Then, the series in (2.10) is summed at $y = 0$ to obtain $\hat{u}(0^{\pm})$, $\hat{w}(0^{\pm})$ and $\hat{\phi}(0^{\pm})$. The series is adjusted by a constant factor to ensure continuity of \hat{w} , and a new guess for σ is then found by substituting these values into (2.9). This guess is considered converged if it differs by less than 10^{-25} from the previous iterate. This simple procedure converges rapidly over the whole parameter space investigated.

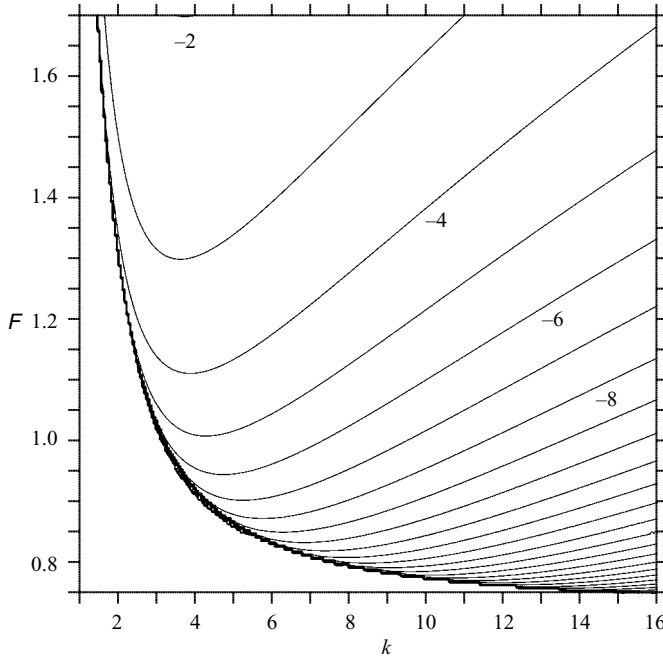


FIGURE 2. Growth rate contours – of $\log_{10}(\sigma_i)$ – in the (k, F) -parameter plane. A few contours are labelled (the contour interval is 1).

When $R \geq 1$, we can demonstrate that the series on the shallow side ($y > 0$) diverges near the front. In this regime, Boss *et al.* (1996) solved the linear equations for all $y > 0$ by numerical integration (using the variable $z = e^{-y/c_+}$). However, the series converges fast for sufficiently large y , and hence it is necessary to perform a numerical integration only over a small range in y typically. The series diverges when the local Rossby number $Re^{-y/c_+} \geq 1$. Convergence slows as $R \rightarrow 1$, so for $R > 0.9$, we use the series only beyond $y = Y_+$, with Y_+ chosen so that the local Rossby number there equals 0.7. From $y = Y_+$ to 0, numerical integration using a fourth-order Runge–Kutta method is used, with a step size $\Delta y \leq 10^{-4} c_+ / \max(k, 1)$. These numerical values ensure both solution methods (series only and series plus numerical integration) at $R = 0.9$ agree to at least 7 digits in their value of σ over $0 < k < 10$. The results reported are insensitive to these parameters except for very small $R - 1$, when growth rates can be of the order of the machine precision.

3. Numerical results

We start by examining the general stability properties over the complete parameter space investigated, namely $1 \leq k \leq 16$ and $0.75 \leq F \leq 1.7$ (or, alternatively $1.08 \leq R \leq 3.68$). Figure 2 shows the growth rate

$$\sigma_i = \text{Im}(\sigma),$$

in logarithmic scale, in the (k, F) -plane. First of all, there is a long-wave cutoff: only wavenumbers larger than some cutoff wavenumber k_c are unstable according to the numerics (we cannot discount other modes of instability, but an extensive search has not revealed any). The long-wave cutoff increases rapidly as $F \rightarrow 1/\sqrt{2}$ or $R \rightarrow 1$, and the growth rates fall sharply. Even quadruple precision is not enough to capture the

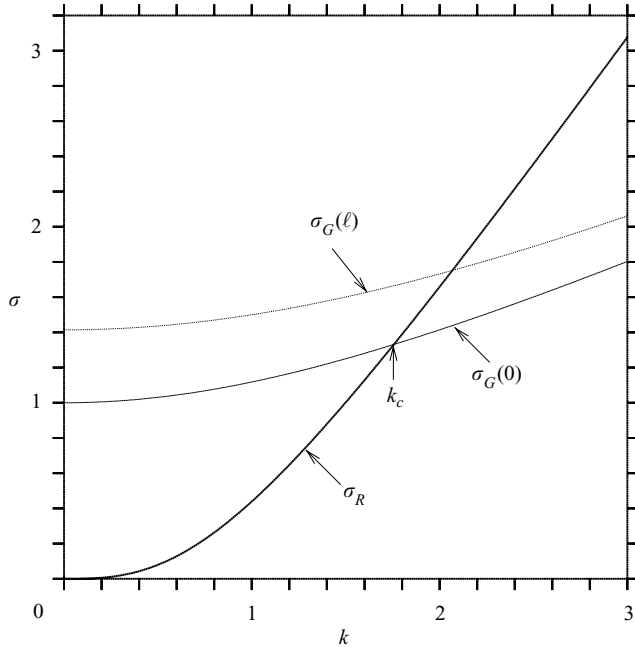


FIGURE 3. Approximate Rossby-wave σ_R and gravity-wave σ_G dispersion relations as a function of x wavenumber k . σ_R makes use of the quasi-geostrophic (QG) approximation (strictly valid for $R \ll 1$), while σ_G applies far from the front on the shallow side. $\sigma_G(0)$ is for a y wavenumber $\ell = 0$, while $\sigma_G(\ell)$ is for finite ℓ (here 2). The curves were generated using $R = 3$ ($F = 1.5$). The predicted long-wave cutoff occurs when $\sigma_R = \sigma_G(0)$, at $k = k_c$. For all $k > k_c$, there exists a real value of ℓ for which $\sigma_R = \sigma_G(\ell)$, and there is always instability. Note: the curves cross only when $R > 1$.

exceedingly weak growth rates below $F = 0.73$, but their existence is clear from the asymptotic analysis described in the next section. Remarkably, growth rates are very small, indeed never greater than 7.76×10^{-6} , when $F \leq 1$. These instabilities would be virtually impossible to detect in numerical simulations of the full equations, and would probably be overwhelmed by nonlinear effects (see discussion).

Instability occurs when the frequency of the (PV-controlled) Rossby wave which, in the first instance, we may approximate by the quasi-geostrophic result (2.15), $\sigma_R = kF[1 - (1 + k^2)^{-1/2}]$, matches (or nearly matches) that of an inertia-gravity wave on the shallow side of the front, $\sigma_G = (1 + c_+^2(k^2 + \ell^2))^{1/2}$, where ℓ is the y wavenumber far from the front (see figure 3). When $F < 1/\sqrt{2}$, no real values of k and ℓ can be found for which $\sigma_R = \sigma_G$. However, for $F \geq 1/\sqrt{2}$, there exists a range of k extending from a long-wave cutoff k_c to ∞ and real values of $\ell(k)$ with matching frequencies. Moreover, there is then a turning point where the phase speed σ/k matches the Doppler-shifted inertia-gravity-wave speed $\bar{\phi}^{1/2} + \bar{u}$. Through this turning point, the character of the mode changes from an evanescent Rossby mode to an oscillatory (but slowly decaying) inertia-gravity mode; this is discussed further in §4.

This qualitative picture is consistent with the numerical results, as demonstrated next. The long-wave cutoff k_c is found to coincide with $\ell = 0$, i.e. an infinitely long wave in y (or equivalently a turning point for $y \rightarrow \infty$). Using $\sigma_R = \sigma_G$ for $\ell = 0$ then gives a relation between k_c and either R or F . This is compared in figure 4 against the numerical results (i.e. the ‘exact’ or full stability analysis). The agreement is

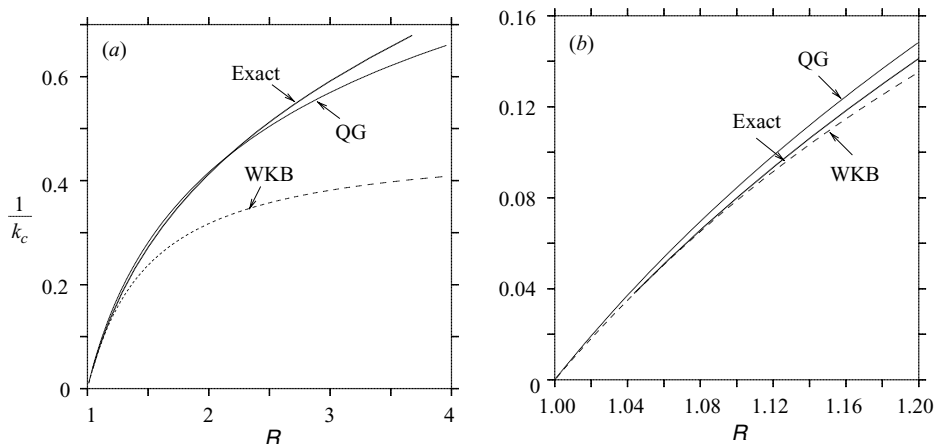


FIGURE 4. Approximate quasi-geostrophic (QG) estimate of the long-wave cutoff (or its inverse $1/k_c$) compared with the ‘exact’ stability analysis for Rossby numbers (a) $1 \leq R \leq 4$ and (b) an enlargement for $1 \leq R \leq 1.2$. Note that the QG model is strictly valid only when $R \ll 1$ ($F \ll 1/\sqrt{2}$). Also shown is the WKB estimate, (4.11) of §4.

spectacular considering that the QG approximation is being pushed well beyond its expected limits of validity. In fact, the agreement remains good for R as large as 3. As $R \rightarrow 1$, the predicted k_c is given by $1/(R-1)$. This result is very close to the exact value: the asymptotic calculations of §4 show that the exact value tends to $k_c = 1.0606/(R-1)$.

The agreement also confirms the simple idea that instability can be predicted by matching the frequencies of Rossby and gravity modes. This idea can also be used to estimate the dependence of the y wavenumber ℓ on k , shown in figure 5 for the case $F=1$ ($R=1.618\dots$). Here, the ‘exact’ results are obtained from the imaginary part of K^+ , see (2.10), since for $y \gg 1$, the disturbance $(\hat{u}, \hat{w}, \hat{\phi}) \sim (\hat{u}_0^+, \hat{w}_0^+, \hat{\phi}_0^+)e^{-K^+y}$. Hence, $\ell = -\text{Im}(K^+)$. The agreement is excellent, apart from a small offset in k . Again this shows that mode matching explains the essential nature of the instability.

The real part of K^+ must be positive for the disturbance to decay as $y \rightarrow \infty$. This is shown for the same case ($F=1$) in figure 6, now using the full stability analysis. Instability erupts at $k=k_c=3.21445\dots$, and for the entire unstable range $K_r^+ > 0$. However, K_r^+ is $O(\sigma_i) \ll 1$, so the decay is extremely slow (but nonetheless exponential). Over any reasonable distance, the disturbance on the shallow side of the front looks like a pure gravity wave (and that on the deep side looks like a decaying Rossby wave). As k passes downward through k_c , it appears that K_r^+ jumps to a large value. Closer inspection (figure 6b) reveals that K_r^+ is in fact continuous, as is the y wavenumber $\ell = -K_i^+$ (this is zero for $k \leq k_c$). Hence, there is a continuous transition to instability, and all unstable modes are found to have finite energy.

4. WKB analysis

The instability of the PV-front can be studied asymptotically in the limit of large wavenumber $k \gg 1$ using a WKB analysis. Ford (1993, 1994) carried out such an analysis and found the growth rates to be exponentially small in k . This type of result is most useful when the largest growth rates occur for $k \gg 1$. This is the case here in the marginally unstable regime $R-1 \ll 1$. However, even in this regime, the

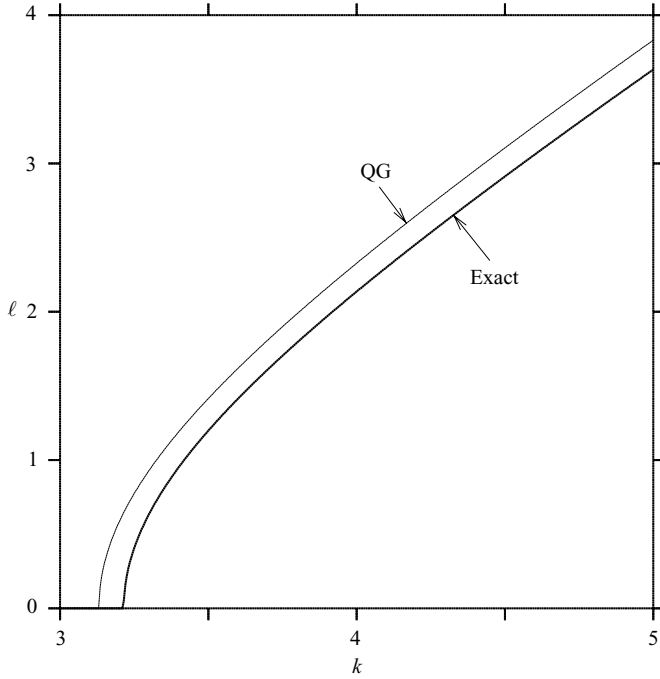


FIGURE 5. Approximate quasi-geostrophic (QG) estimate of the y wavenumber ℓ compared with the ‘exact’ stability analysis as a function of k . Here, we have taken $F = 1$ ($R = 1.618\dots$). Maximum instability is observed for $k = 4.3386$. Note that ℓ is real only for $k \geq k_c$.

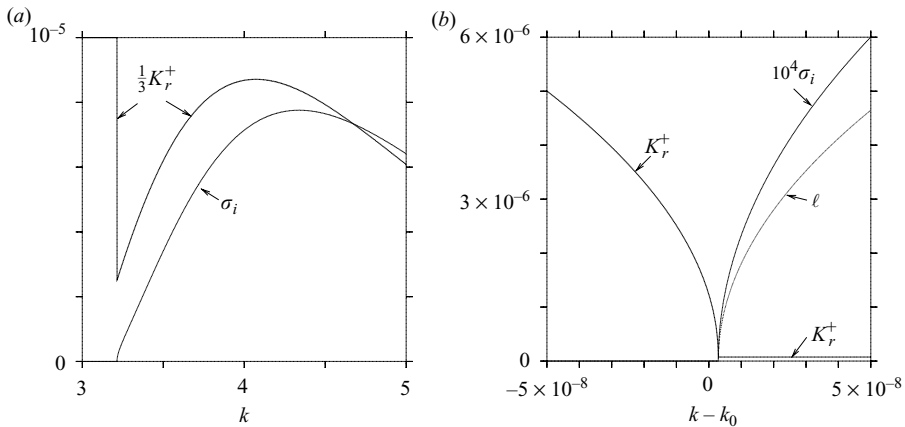


FIGURE 6. Dependence of K_r^+ (the real part of K^+) and the growth rate σ_i on k for $F = 1$ ($R = 1.618\dots$). (a) A range of k including the long-wave cutoff and the peak instability. (b) Centred on $k_0 = 3.21445147$, focuses on the region of the long-wave cutoff; it also shows $\ell = -K_i^+$.

maximum growth rate cannot be inferred directly from Ford’s (1993) work. This is because Ford’s analysis breaks down for k near k_c , where the maximum growth rate is attained. We therefore extend the WKB analysis to obtain an approximation valid in this region. We only sketch the derivation and relegate details to the Appendix.

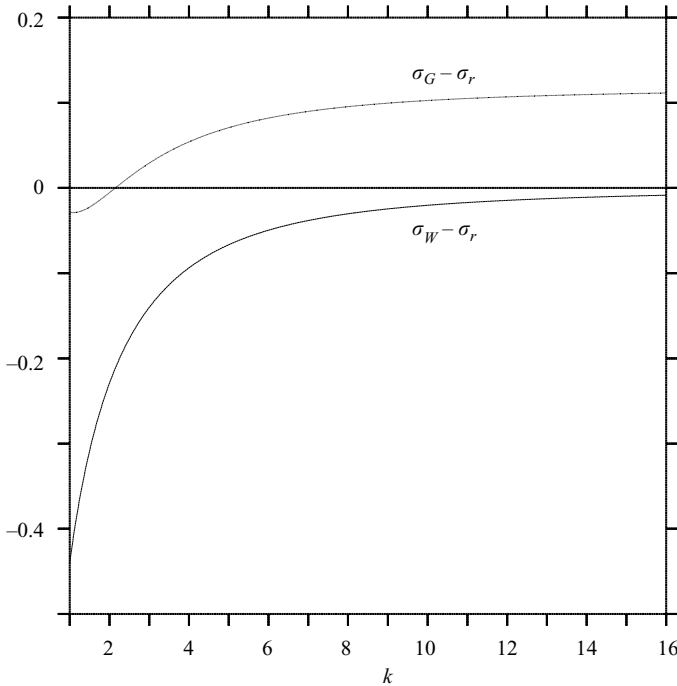


FIGURE 7. Difference between the WKB frequency and the ‘exact’ frequency, $\sigma_W - \sigma_r$, and between the QG frequency and the ‘exact’ frequency, $\sigma_G - \sigma_r$, for $F = 1$. Note $\sigma_r = 14.89 \dots$ for $k = 16$.

The WKB approach seeks solutions for $k \gg 1$ in the form

$$\hat{u} = \mathcal{A}(y)e^{k\Psi(y)}, \tag{4.1}$$

and expands \mathcal{A} , Ψ , and the phase speed c in inverse powers of k . In Appendix A we obtain in this manner the approximation

$$\sigma = kF - F(1 + F^2/4)^{1/2} + O(1/k^2), \tag{4.2}$$

for the (Rossby-wave) frequency, consistent with the quasi-geostrophic approximation (2.15) in the limit $F \rightarrow 0$. Note that this approximation is one order more accurate than that obtained by Ford (1993), although the new, $O(1/k)$ term turns out to vanish. This higher accuracy is required below for the computation of the growth rate near k_c . Figure 7 shows how the estimate (4.2), denoted σ_W in the figure, compares with the actual real part of σ , for $F = 1$ and over the same range of wavenumbers used in figure 2. A comparison with the QG estimate (denoted σ_G) is also given. At large k , the WKB estimate is more accurate, as would be expected.

The computation leading to (4.2) can, in principle, be extended to obtain approximations to σ accurate to higher orders $O(1/k^n)$. To all algebraic orders, σ is real, because, as recognized by Ford (1993; see also Knessl & Keller 1992), the instability is characterized by a non-zero imaginary part of σ that is exponentially small in k . This can be traced to the existence, ignored in the above developments, of a turning point where Ψ changes from being purely real to being purely imaginary. To leading order, the turning-point position, y_* say, is determined by the condition

$$\bar{c}_0^2(y_*) = [F - \bar{u}(y_*)]^2 = \bar{\phi}(y_*), \tag{4.3}$$

which has a solution $y_* > 0$ for $R > 1$. This condition expresses the match at y_* between the leading-order Rossby-wave frequency kF and the Doppler-shifted gravity-wave frequency $k(\bar{u} + \bar{\phi}^{1/2})$.

The instability growth rate $\sigma_i = \text{Im}(\sigma)$ can be estimated by modifying the WKB solution for $y > 0$ to account for the existence of this turning point. Briefly, the decaying solution (4.1) must be supplemented by an exponentially growing solution, which is subdominant for $0 < y < y_*$, but becomes of a similar order as the decaying solution for $y \approx y_*$. The combination of the two solutions, with appropriate relative amplitudes, ensures that the oscillatory solution for $y > y_*$ satisfies a radiation boundary condition as $y \rightarrow \infty$. Note that the exponential decay of the solutions for $y \rightarrow \infty$ with decay rate proportional to σ_i described in §3 is not apparent in the solution for \hat{u} so obtained; this is because the WKB analysis implicitly assumes that $y \ll \sigma_i^{-1}$.

In the Appendix, we derive several approximations to σ_i , valid in regions of parameter space distinguished by the relative values of $k \gg 1$ and

$$\delta \equiv R - 1 > 0.$$

Assuming that $k \gg 1$, we obtain the expression

$$\sigma_i \sim F \frac{c_+ + c_-}{4} e^{-2k\Psi_*} \quad \text{where} \quad \Psi_* = \int_0^{y_*} \left(1 - \frac{\bar{c}^2}{\bar{\phi}}\right)^{1/2} dy, \tag{4.4}$$

and c is approximated as $c \sim F + c_1/k$, with $c_1 = -F(1 + F^2)^{1/2}$. This result is valid uniformly for $k \gg 1$ in two regimes: (i) $\delta = O(1)$, and (ii) $\delta = O(1/k) \ll 1$. In regime (i), Ψ_* can be expanded in inverse powers of k to find

$$\Psi_* = \int_0^{y_*} \left(1 - \frac{\bar{c}_0^2}{\bar{\phi}}\right)^{1/2} dy - \frac{1}{k} \int_0^{y_*} \frac{\bar{c}_0 c_1}{\bar{\phi} (1 - \bar{c}_0^2/\bar{\phi})^{1/2}} dy + O(1/k^2). \tag{4.5}$$

This is the result originally obtained by Ford (1993). It fails in regime (ii) because for sufficiently large $y < y_*$, $1 - \bar{c}_0^2/\bar{\phi} = O(\delta)$ and the expansion leading to (4.5) becomes disordered.

In regime (ii), since $k\delta = O(1)$, it is natural to introduce the scaled dimensionless wavenumber

$$\kappa \equiv \delta c_+ k = O(1). \tag{4.6}$$

Expanding (4.3), it emerges that there is a turning point and hence instability is possible provided that $\kappa > 3/4$. Taking (4.6) into account, this gives a first approximation to the cutoff wavenumber

$$k_c = \frac{3\sqrt{2}}{4\delta} + O(1). \tag{4.7}$$

This result is refined below with the calculation of the $O(1)$ term. Assuming $k > k_c$, i.e. $\kappa > 3/4$, the growth rate in regime (ii) is found as

$$\sigma_i \sim \frac{3}{8} e^{-2k\Psi_*}, \tag{4.8}$$

where Ψ_* is expanded in powers of δ according to

$$2k\Psi_* = \frac{a_1}{\delta} + \frac{a_2}{\delta^{1/2}} + a_3 + O(\delta^{1/2}). \tag{4.9}$$

Here a_1 , a_2 and a_3 are functions of κ defined by integrals and given by $a_1 = 5.782\kappa$, $a_2 = -\pi[2\kappa(4\kappa - 3)]^{1/2}$, and $a_3 = 7.052\kappa - 3.789$. (See (A 12)–(A 14) for the exact expressions of the numerical constants given here to four-digit accuracy.)

As the expression for a_2 suggests, the approximation (4.8)–(4.9) breaks down for $\kappa - 3/4 = O(\delta)$, i.e. in the vicinity of the cutoff wavenumber. Since this is also where the maximum of σ_i is attained, it is important to derive an asymptotic formula appropriate for this regime, which we denote by (iii) and is defined in dimensional terms by $k - 3\sqrt{2}/(4\delta) = O(1)$.

Calculations detailed in the Appendix examine the instability in regime (iii). There we start by defining the scaled wavenumber

$$k \equiv \frac{\kappa - 3/4}{\delta} = O(1) \tag{4.10}$$

and show that instability occurs only for $k > 31/24$. This provides the estimate of the cutoff wavenumber

$$k_c = \frac{3\sqrt{2}}{4\delta} + \frac{71\sqrt{2}}{48} + O(\delta), \tag{4.11}$$

which improves on (4.7). (This estimate is compared with the numerical and quasi-geostrophic results in figure 4.) The instability growth rate is then found in the form

$$\sigma_i \sim \frac{3}{4} \sinh(\pi\nu) e^{-2\Phi_*/\delta}, \tag{4.12}$$

where

$$\nu = (6k - 31/4)^{1/2}, \quad 2\Phi_* = b_1 + \delta b_2 + O(\delta^2).$$

Here, $b_1 = 4.336$ and $b_2 = 5.782k + 1.5$ are defined by integrals given in (A 23). Note that the two approximations (4.8) and (4.12) can be verified to match in the intermediate region $\delta \ll |\kappa - 3/4| \ll 1$ where both are valid.

The expression (4.12) is the sought approximation for the growth rate valid for k in an $O(1)$ neighbourhood of the cutoff wavenumber (4.11). It allows the estimation of the maximum growth rate, found to be achieved for $k = 1.735$. It also shows how the sharp drop in the growth rate as the long-wave cutoff is approached is described by a hyperbolic sine function. Mathematically, this behaviour arises because the standard Airy-function connection across a turning point is replaced near the cutoff by a Bessel-function connection; it is likely to be generic for problems where marginal stability coincides with the appearance of a turning point at infinity. Note that the crude approximation $\exp(-b_1/\delta)$ of the maximum growth rate can be inferred from Ford's (1993) result; the full expression (4.12) shows this overestimates the growth rate by more than three orders of magnitude.

The three growth-rate estimates (4.4), (4.8) and (4.12), denoted by WKB_1 , WKB_2 and WKB_3 , respectively, are compared in figure 8 with the 'exact' growth rate as a function of κ and for two values of F . For the case $F = 0.8$ ($\delta = R - 1 = 0.1816\dots$, figure 8*a, b*), the estimate (4.4) accurately captures the exponential decay of σ_i for large κ . It also performs reasonably well for $\kappa = O(1)$ and away from the long-wave cutoff; this is the range for which it reduces to (4.8). The performance of the third estimate (4.12) is best appreciated from figure 8*(b)* which focuses on the region near the cutoff wavenumber and maximum growth rate. The estimate provides a good approximation to the cutoff wavenumber, but only a crude one for the behaviour of σ_i near its maximum. The situation improves as δ decreases. This is apparent from the results obtained for $F = 0.76$ ($\delta = 0.1018\dots$) shown in figure 8*(c)*. This time, (4.12) estimates well the long-wave cutoff ($\kappa_c = 0.88152\dots$ versus $0.86513\dots$) and

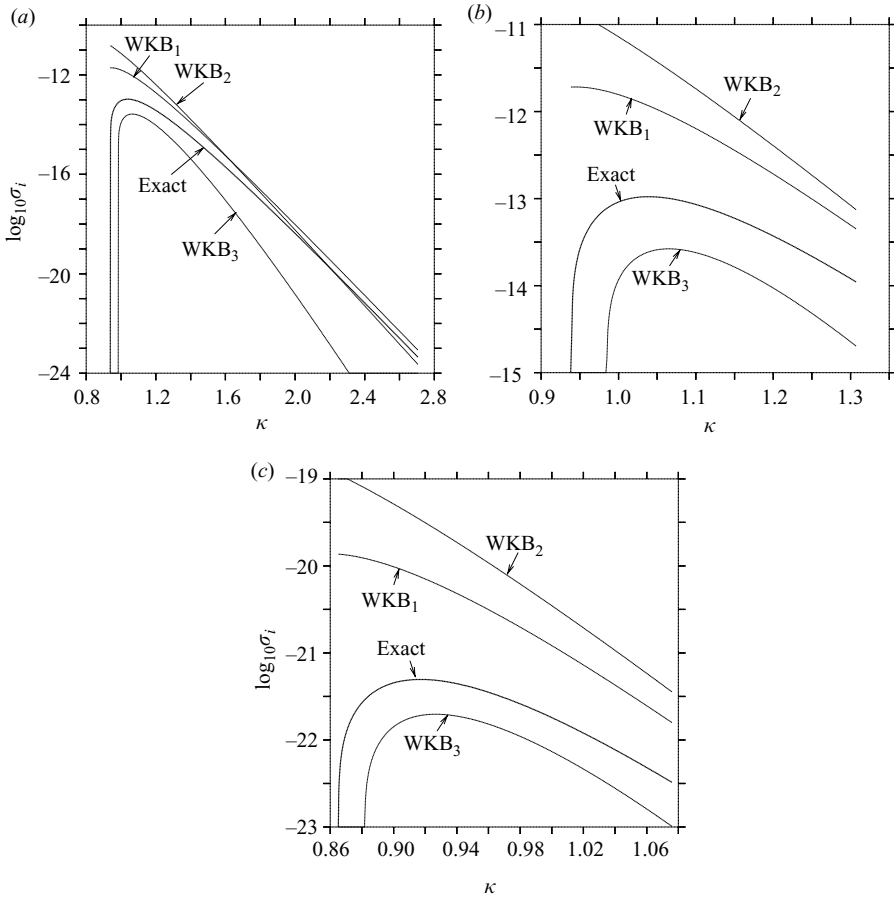


FIGURE 8. Comparison of various WKB estimates of the growth rate σ_i with the ‘exact’ result, as a function of the scaled wavenumber κ , for $F=0.8$ ($R=1.1816\dots$) (a, b) and $F=0.76$ ($R=1.1018\dots$) (c). The labels WKB₁, WKB₂ and WKB₃ correspond to formulae (4.4), (4.8) and (4.12), respectively.

provides the growth rate within a factor of approximately four. Convergence of (4.12) to the exact values of σ_i with $\delta \rightarrow 0$ appears to be very slow, and the limitations of quadruple precision forbid using a value of δ small enough to demonstrate it plainly. The slow convergence is illustrated well by the fact that values as small as $\delta = 10^{-9}$ are necessary to observe the overlap between (4.4) (or (4.8)) and (4.12) in their region of common asymptotic validity.

5. Discussion

This paper has re-examined the stability of a potential-vorticity front in the rotating shallow-water equations. This was previously examined by Ford (1993) and Boss *et al.* (1996). The front, whose properties depend on a single dimensionless parameter F (or equivalently R), is unstable provided that $F > 1/\sqrt{2}$ (or $R > 1$). The associated disturbances have Rossby-wave characteristics near the front and inertia-gravity-wave characteristics far away, on the shallow side of the front. The instability is exceptionally weak, with growth rates scaling exponentially in $1/(R-1)$ as $R \rightarrow 1$ and

numerically very small even for $R - 1 = O(1)$. Correspondingly, the amplitude of the growing mode decays very slowly with the distance from the front.

The characteristics of the instability (y -wavenumber ℓ , cutoff wavenumber k_c) can be inferred from the condition of frequency matching between the near-front Rossby wave, and the far-field inertia-gravity wave. Whilst the dispersion relation of the latter is given explicitly (since the basic-flow height is constant in the far field), the Rossby-wave dispersion relation must be approximated in some way. The quasi-geostrophic approximation (2.15), which formally assumes $R \ll 1$, turns out to be useful for this purpose since it proves remarkably accurate well into the unstable regime $R > 1$. An asymptotically consistent alternative is provided by the WKB approximation (4.2) valid for $k \gg 1$. This approximation makes it possible to describe analytically the instability, including the long-wave cutoff, in the limit $R \rightarrow 1$, when all the unstable wavenumbers satisfy $k = O(1/(R - 1)) \gg 1$.

The instability studied in this paper illustrates several aspects of the concept of balance for rapidly rotating fluids. First, the stability in the regime $0 < R < 1$, established by Ford (1993) using Ripa's (1983) theorem, is consistent with the idea that a balanced flow, represented here by the frontal Rossby wave, is isolated from the inertia-gravity waves when a complete frequency separation exists. Secondly, the threshold value $R = 1$ for instability coincides precisely with the breakdown of the frequency separation. Thirdly, the unbalanced phenomenon – here the instability – is exponentially weak in the limit of small frequency overlap, and turns out to remain weak over a wide parameter range. This last point echoes what is frequently observed in more realistic flows, namely the weakness of unbalanced phenomena even in the absence of a frequency separation.

Note that analogous instabilities in continuously stratified flows do not appear to have a threshold Rossby number (e.g. Yavneh *et al.* 2001; McWilliams *et al.* 2004). This can be attributed to the dispersion relation for inertia-gravity waves in stratified flows which allows for arbitrarily small phase speed (for large wavenumbers) and hence for a match with any flow speed. (This can be rephrased by stating that there is no analogue of Ripa's (1983) theorem for continuously stratified flows.)

In realistic nonlinear time-dependent flows, there is of course no frequency separation, and the excitation of inertia-gravity waves can be expected to take place for all values of R . The mechanism for this excitation can be either an instability of a type generalizing that studied in this paper, or a spontaneous-adjustment mechanism similar to that examined by Vanneste & Yavneh (2004) and Vanneste (2004) in simple toy models. In a more realistic context, the nonlinear evolution of a potential-vorticity front in the full shallow-water equations would seem an ideal problem to examine this excitation. However, based on preliminary numerical simulations, even this simplest of problems appears formidable, requiring highly accurate numerics as well as a careful initialization to avoid a significant presence of inertia-gravity waves from the start. For small R , it is practically impossible to differentiate inertia-gravity waves from numerical error, even when numerical methods specifically designed to handle potential vorticity discontinuities are employed (cf. Mohebalhojeh & Dritschel 2001 and references). For larger R , it becomes increasingly difficult to separate balanced and unbalanced motions, and for $R > \sqrt{2}$, the governing equations themselves may break down as a result of shock formation (thereby violating the underlying hydrostatic approximation). In a way, these difficulties emphasize the tight control often exerted by balanced motions in geophysical flows.

We conclude by remarking on the assumption of a discontinuous PV distribution made for the basic state. This is motivated not only by the (relative) simplicity of the

analysis, but also by the observation that geophysical flows often exhibit sharp PV fronts separating well-homogenized regions. Smoothing the PV distribution across the front would lead to significant changes, with the appearance of a critical layer for the Rossby wave. However, because the non-zero imaginary part of σ means there is no singularity at the critical level, we can expect the instabilities found in this paper to remain relevant. This is also suggested by the results of McWilliams *et al.* (2004) where mixed-mode (Kelvin/inertia-gravity) instabilities appear in the presence of a smooth PV gradient and hence of critical levels. Given the very small values of σ_i found here, it would nevertheless be of interest to assess the effect of smoothing the PV front, and analyse the interplay between the growth rate and the sharpness of the PV front.

J. V. is funded by a NERC Advanced Research Fellowship.

Appendix. WKB derivation

In this Appendix, we provide some details of the derivation of the approximations (4.2) for the Rossby-wave frequency, and (4.4), (4.8) and (4.12) for the instability growth rate.

A.1. Rossby-wave frequency

From (2.6)–(2.8), a single equation for \hat{u} can be derived. This reads

$$\hat{u}'' + \frac{\bar{\phi}'}{\bar{\phi}}\hat{u}' + \left[k^2 \left(\frac{\bar{c}^2}{\bar{\phi}} - 1 \right) - \frac{\bar{\phi}}{c_{\pm}^4} + \frac{(\bar{\phi}\bar{c}')}{c_{\pm}^2\bar{\phi}} \right] \hat{u} = O(1/k), \quad (\text{A } 1)$$

where $\bar{c} = c - \bar{u} = \sigma/k - \bar{u}$. The associated boundary condition is given in (2.9), with

$$\hat{v} = -i(\hat{u}' + \bar{c}\hat{u})/k + O(1/k^2). \quad (\text{A } 2)$$

Note that the error terms assume that $\hat{u}'/\hat{u} = O(k)$, as is relevant for the WKB solution.

We seek solutions to (A 1) for $k \gg 1$ in the form (4.1) and require that both (\mathcal{A}, Ψ) and $(\mathcal{A}, -\Psi)$ be solutions. This leads to

$$k^2\Psi'^2\mathcal{A} + \mathcal{A}'' + \frac{\bar{\phi}'}{\bar{\phi}}\mathcal{A}' + \left[k^2 \left(\frac{\bar{c}^2}{\bar{\phi}} - 1 \right) - \frac{\bar{\phi}}{c_{\pm}^4} + \frac{(\bar{\phi}\bar{c}')}{c_{\pm}^2\bar{\phi}} \right] \mathcal{A} = O(1/k), \quad (\text{A } 3)$$

$$2k\Psi'\mathcal{A}' + k\Psi''\mathcal{A} + k\frac{\bar{\phi}'}{\bar{\phi}}\Psi'\mathcal{A} = O(1/k). \quad (\text{A } 4)$$

Equation (A 3) can be solved perturbatively, by expanding

$$\Psi = \Psi_0 + \Psi_1/k + O(1/k^2), \quad c = F + c_1/k + c_2/k^2 + O(1/k^3). \quad (\text{A } 5)$$

This gives

$$\Psi_0' = \mp \left(1 - \frac{\bar{c}_0^2}{\bar{\phi}} \right)^{1/2}, \quad \Psi_1' = -\frac{\bar{c}_0 c_1}{\bar{\phi}(1 - \bar{c}_0^2/\bar{\phi})^{1/2}}, \quad (\text{A } 6)$$

where the signs, corresponding to $y \gtrless 0$, are chosen to ensure exponential decay away from the PV front. At $y = 0^{\pm}$, in particular, $\Psi_0' = \mp 1$, $\Psi_0'' = 0$ and $\Psi_1' = 0$. It follows from (A 2), (4.1) and (A 4) that

$$\hat{v} = -i \left[\Psi_0' + \frac{1}{k}\Psi_1' - \frac{1}{2k} \left(\frac{\Psi_0''}{\Psi_0'} + \frac{\bar{\phi}'}{\bar{\phi}} \right) + \frac{\bar{c}_0}{kc_{\pm}^2} \right] \hat{u} + O(1/k^2),$$

and hence that

$$\hat{v} = i \left(\pm 1 - \frac{F}{2k} \right) \hat{u} + O(1/k^2) \quad \text{at } y = 0^\pm. \tag{A 7}$$

Introducing (A 7) into the jump condition (2.9) leads to the first two corrections in the asymptotic expansion of c , namely $c_1 = -F(c_+ + c_-)/2$ and $c_2 = 0$, and hence to the approximation (4.2) for the frequency.

A.2. Growth rate in regimes (i) and (ii)

As described in §4, the instability is associated with the existence of a turning point y_* satisfying (4.3). Because of this turning point, we must consider a superposition of growing and decaying WKB solutions for $y > 0$. In regime (i), defined by $k(R - 1) \gg 1$ and considered by Ford (1993), the WKB solution corresponding to (A 6) remains valid for $y_* - y \gg k^{-2/3}$. Taking

$$\Psi_0 = \int_0^y \left(1 - \frac{\bar{c}_0^2(y')}{\bar{\phi}(y')} \right)^{1/2} dy', \quad \Psi_1 = - \int_0^y \frac{\bar{c}_0(y')c_1}{\bar{\phi}(y')[1 - \bar{c}_0^2(y')/\bar{\phi}(y')]^{1/2}} dy',$$

the solution in this range is written as $u = \mathcal{A}(y)[a \exp(k\Psi_0 + \Psi_1) + b \exp(-k\Psi_0 - \Psi_1)] + O(1/k)$, for two constants a and b . The Airy-function connection with an outward-propagating solution for $y - y_* \gg k^{-2/3}$ imposes the relationship

$$\frac{a e^{k\Psi_0(y_*) + \Psi_1(y_*)}}{b e^{-k\Psi_0(y_*) - \Psi_1(y_*)}} = \frac{i}{2} \tag{A 8}$$

(Ford 1993, 1994; see also, e.g., Bender & Orszag 1978, §10). It then follows from (A 2) that $\hat{v} = -i\Psi'_0(a \exp(k\Psi_0 + \Psi_1) - b \exp(-k\Psi_0 - \Psi_1)) + O(1/k)$ and

$$c_1 \left(1 + \frac{1 + a/b}{1 - a/b} \right) = -F(c_+ + c_-). \tag{A 9}$$

Using (A 8) and the smallness of the exponentials gives

$$\text{Im } c_1 = \frac{F(c_+ + c_-)}{4} \exp(-2[k\Psi_0(y_*) + \Psi_1(y_*)]), \tag{A 10}$$

that is, the estimate (4.4) for the growth rate, with Ψ_* approximated as in (4.5).

The WKB approach just outlined breaks down when $k(R - 1) = O(1)$. To estimate the growth rate in this regime, denoted by (ii), a different WKB expansion can be used, with $\delta = R - 1$ as the small parameter. An alternative is to note that the derivation above remains valid in regime (ii) provided that we avoid introducing the expansion

$$\frac{\bar{c}^2}{\bar{\phi}} - 1 = \frac{\bar{c}_0^2}{\bar{\phi}} - 1 + \frac{2\bar{c}_0 c_1}{k\bar{\phi}} + O(1/k^2).$$

Repeating the derivation leading to (A 10), but without this expansion (and with c approximated according to (4.2)) leads to the growth rate in the form (4.4). This approximation is valid uniformly for both regimes (i) $\delta = O(1)$ and (ii) $\delta = O(1/k)$. The simplified expression (4.5) follows in regime (i) by expansion in inverse powers of k . We now derive a simplified expression valid in regime (ii) by expansion in powers of δ .

Introducing the scaled wavenumber κ defined in (4.6) and noting that $F = 1/\sqrt{2} + 3/(4\sqrt{2})\delta + O(\delta^2)$, we expand equation (4.3) for the turning-point position and obtain

$$z_* \equiv e^{-y_*/c_+} = \frac{2\delta}{3} \left(1 - \frac{3}{4\kappa} \right) + O(\delta^2). \tag{A 11}$$

The prefactor $F(c_+ + c_-)/4$ in (4.4) reduces to $3/8$ (cf. (4.8)). Then, using (4.6) and (A 11), we compute

$$\frac{c}{c_+} = 1 + \delta \left(1 - \frac{3}{4\kappa} \right) + O(\delta^2)$$

and

$$1 - \frac{\bar{c}^2}{\bar{\phi}} = \frac{z(3-z)}{1+z} - \delta \frac{2(1-z^2)(1-z-3/(4\kappa)) - z(1-z)^2}{(1+z)^2} + O(\delta^2),$$

where $z \equiv \exp(-y/c_+)$. Substituting into (A 4) and changing the variable of integration from y to z leads to an integral that is best expanded by splitting the integration range $[z_*, 1]$ at some intermediate $z_* \ll z \ll 1$ (e.g. Hinch 1991, §3.4). A tedious but straightforward computation leads to the approximation (4.9), where

$$a_1 = 2\kappa \int_0^1 \left(\frac{3-z}{z(1+z)} \right)^{1/2} dz, \tag{A 12}$$

$$a_2 = -\pi[2\kappa(4\kappa - 3)]^{1/2}, \tag{A 13}$$

$$a_3 = \kappa \left[\frac{4}{\sqrt{3}} - \int_0^1 \left(\frac{(z-1)(z^2+z-2)}{z^{3/2}(1+z)^{3/2}(3-z)^{1/2}} - \frac{2}{\sqrt{3}z^{3/2}} \right) dz \right] - \frac{3}{\sqrt{3}} - \frac{3}{2} \int_0^1 \left(\frac{z^2-1}{z^{3/2}(1+z)^{3/2}(3-z)^{1/2}} + \frac{1}{\sqrt{3}z^{3/2}} \right) dz. \tag{A 14}$$

The same result obtains if (A 1) is expanded for $\delta \ll 1$, $\kappa = O(1)$, and a WKB-analysis of the resulting equation is performed. The derivation is then particularly tedious because approximations in four distinct regions ($z = O(1)$, $z_* < z = O(\delta)$, $|z - z_*| = O(\delta^{1/3})$, and $z < z_*$) must be matched.

A.3. Growth rate in regime (iii)

In an $O(\delta)$ neighbourhood of the cutoff wavenumber $\kappa = 3/4 + O(\delta)$, the WKB approximation used above breaks down. This is because for $y \gg 1$, $k^2(1 - \bar{c}_0^2/\bar{\phi}) = O(1)$, and the other terms multiplying \hat{u} in (A 1) must be taken into account. The turning point satisfies $z_* = O(\delta^2)$ and two expansions must be derived and matched: the first one valid for $z = O(1)$, the second valid for $z = O(\delta^2)$.

Let us first consider the region $z = O(1)$. Introducing (4.6) and $\kappa = 3/4 + \delta k$ reduces (A 1) to

$$\hat{u}'' + \frac{\bar{\phi}'}{\bar{\phi}} \hat{u}' + \left[\frac{1}{\delta^2} \frac{9z(z-3)}{16(1+z)} + \frac{1}{\delta} \left(\frac{9z}{16} \frac{2(z^2-1) - (1-z)^2}{(1+z)^2} + \frac{3kz(z-3)}{2(1+z)} \right) \right] \hat{u} = O(1).$$

Seeking a solution in the WKB form

$$\hat{u} = \mathcal{A}(y)(ae^{\Phi/\delta} + be^{-\Phi/\delta}) \quad \text{with} \quad \Phi = \Phi_0 + \delta\Phi_1 + O(\delta^2), \tag{A 15}$$

leads to

$$\Phi_0(z) = \frac{3}{4} \int_z^1 \left[\frac{3-z'}{z'(1+z')} \right]^{1/2} dz', \tag{A 16}$$

$$\Phi_1(z) = k \int_z^1 \left[\frac{3-z'}{z'(1+z')} \right]^{1/2} dz' + \frac{3}{8} \int_z^1 \frac{2(1-z'^2) + (1-z')^2}{z'^{1/2}(1+z')^{3/2}(3-z')^{1/2}} dz'. \tag{A 17}$$

The behaviour for $\delta^2 \ll z \ll 1$, required for matching, readily follows as

$$\Phi = \Phi_* - \frac{3\sqrt{3}z}{2} + O(z, \delta^2) \quad \text{with} \quad \Phi_* = \Phi_0(0) + \delta\Phi_1(0). \tag{A 18}$$

We next consider the region $z = O(\delta^2)$. Because $k^2 (\bar{c}^2/\bar{\phi} - 1) = O(1)$ for $z = O(\delta^2)$, the $O(1/k)$ -accurate or, equivalently $O(\delta)$ -accurate approximation (4.2) to σ must be used, and the $O(1)$ terms in (A 1) must be taken into account. Defining the $O(1)$ scaled variable ζ by

$$z = \delta^2 \zeta,$$

we approximate (A 1) as

$$\frac{d^2 \hat{u}}{d\zeta^2} + \frac{1}{\zeta} \frac{d\hat{u}}{d\zeta} - \frac{27(\zeta - \zeta_*)}{16\zeta^2} \hat{u} = O(\delta) \quad \text{where} \quad \zeta_* = \frac{8}{9} \left(k - \frac{31}{24} \right). \quad (\text{A } 19)$$

Clearly, there is a turning point in the domain – and hence an instability – only if $k > 31/24$. This provides the estimate (4.11) for the cutoff wavenumber.

Equation (A 19) can be solved explicitly in terms of modified Bessel functions of imaginary order. In the notation of Dunster (1990), we write the general solution as

$$\hat{u} = \alpha K_{iv}(s) + \beta L_{iv}(s), \quad (\text{A } 20)$$

where α and β are arbitrary constants,

$$s = \frac{3\sqrt{3\zeta}}{2}, \quad v = \frac{3\sqrt{3\zeta_*}}{2} = (6k - 31/4)^{1/2}. \quad (\text{A } 21)$$

A relationship between α and β is obtained by imposing the radiation condition as $y \rightarrow \infty$, i.e. as $s \rightarrow 0$. Using the asymptotics

$$K_{iv}(s) \sim - \left(\frac{\pi}{v \sin(\pi v)} \right)^{1/2} \sin(v \log(s/2) - \varphi),$$

$$L_{iv}(s) \sim \left(\frac{\pi}{v \sin(\pi v)} \right)^{1/2} \cos(v \log(s/2) - \varphi),$$

as $s \rightarrow 0$, with $\varphi = \arg[\Gamma(1 + iv)]$ (Dunster 1990), we obtain from (A 20) the large- y behaviour

$$\hat{u} \sim \left(\frac{\pi}{v \sin(\pi v)} \right)^{1/2} [\alpha \sin(vy/2 + C) + \beta \cos(vy/2 + C)],$$

where C is independent of y . The radiation condition for $y \rightarrow \infty$ then imposes

$$\frac{\alpha}{\beta} = i. \quad (\text{A } 22)$$

We now match (A 20) with (A 18). Using the asymptotics

$$K_{iv}(s) \sim \left(\frac{\pi}{2s} \right)^{1/2} e^{-s}, \quad L_{iv}(s) \sim \frac{1}{\sinh(\pi v)} \left(\frac{\pi}{2s} \right)^{1/2} e^s$$

of the Bessel functions as $s \rightarrow \infty$ (Dunster 1990), we obtain from (A 15), (A 18), (A 20) and (A 22) that

$$\frac{ae^{\Phi_*/\delta}}{be^{-\Phi_*/\delta}} = \sinh(\pi v) \frac{\alpha}{\beta} = i \sinh(\pi v).$$

Applying the jump condition leads to an expression for c_1 similar to (A 9), from which we deduce (4.12). Using (A 16)–(A 18), the exponent $2\Phi_*$ can be written explicitly as $2\Phi_* = b_1 + \delta b_2$, with

$$b_1 = \frac{3}{2} \int_0^1 \left[\frac{3-z}{z(1+z)} \right]^{1/2} dz, \quad b_2 = 2k \int_0^1 \left[\frac{3-z}{z(1+z)} \right]^{1/2} dz + \frac{3}{2}. \quad (\text{A } 23)$$

With ν given in (A 21), this provides a closed form approximation to the instability growth rate near the long-wave cutoff.

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